COMPUTATIONAL FLUID DYNAMICS · J. D. ANDERSON, CH. 3, PP. 111–121

Mathematical Behaviour of PDEs: Parabolic & Elliptic Equations, the Blunt-Body Problem and Well-Posed Problems

Complete study notes. Every topic on the pages is covered, every figure is redrawn, Problems 3.1–3.5 are fully solved, and there's a quiz with an answer key at the end.

WHAT THESE PAGES COVER

0	Background you need (classification, characteristics)
1	§3.4.1 (end): Unsteady inviscid flow, time-marching
2	§3.4.2 Parabolic equations: 2-D, 3-D
2a	Steady boundary-layer flows
2b	"Parabolized" viscous flows (PNS)
2c	Unsteady thermal conduction (Eqs 3.25–3.28)
3	§3.4.3 Elliptic equations, BC types, flows
4	§3.4.4 Supersonic blunt body revisited
5	Mixed nature of the Navier–Stokes equations
6	§3.5 Well-posed problems
7	§3.6 Summary and master table
8	Problems 3.1–3.5, fully solved
9	Quiz (60 questions) and answer key
10	One-page revision sheet

2-HOUR PLAN FOR WRITING THESE NOTES BY HAND

0:00–0:15	§0 Background. Copy the discriminant box and Fig. A.
0:15–0:25	§1 Time-marching. Copy Fig. B.
0:25–0:55	§2 Parabolic. Copy Figs 3.8–3.12 and the derivation of Eq. 3.25 → 3.28.
0:55–1:10	§3 Elliptic. Copy Figs 3.13 and 3.14, the BC types and both flows.
1:10–1:25	§4–§7 Blunt body, mixed NS, well-posed, master table.
1:25–1:50	§8 Problems 3.1–3.5. Work them yourself, then check.
1:50–2:00	§10 Revision sheet, then take the quiz in §9.

Tip: you don't have to copy the "Why it matters" boxes word for word. Do copy every boxed equation and figure.

THE ONE IDEA BEHIND THIS WHOLE CHAPTER

How a PDE is classified (hyperbolic, parabolic or elliptic) tells you **how information travels** in the flow. That decides **how you must solve it** in CFD. Hyperbolic and parabolic equations can be **marched** step by step from known initial data. Elliptic equations must be solved **simultaneously** over the whole closed domain, with boundary conditions on the **entire** boundary.

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0 · Background You Need (from §3.2–§3.4.1)

These pages build on the earlier part of Chapter 3. Here is the minimum you must know.

0.1 Classifying a second-order PDE

For a PDE in two independent variables x, y:

$$A\,\frac{\partial^2\phi}{\partial x^2} + B\,\frac{\partial^2\phi}{\partial x\,\partial y} + C\,\frac{\partial^2\phi}{\partial y^2} + (\text{lower-order terms}) = 0$$

the **characteristic lines** satisfy $A\,(dy)^2 - B\,dx\,dy + C\,(dx)^2 = 0$, i.e.

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}, \qquad D \equiv B^2 - 4AC$$

Discriminant	Type	Characteristics	Example
$D > 0$	**Hyperbolic**	two real, distinct	wave eq., steady supersonic inviscid flow
$D = 0$	**Parabolic**	one real (repeated)	heat eq., boundary-layer eqs.
$D < 0$	**Elliptic**	imaginary (none real)	Laplace eq., steady subsonic inviscid flow

0.2 Cramer's-rule method for a first-order system (§3.2)

Anderson writes a pair of quasi-linear first-order equations in the unknowns u, v:

$$a_1 u_x + b_1 u_y + c_1 v_x + d_1 v_y = f_1, \qquad a_2 u_x + b_2 u_y + c_2 v_x + d_2 v_y = f_2$$

Add $du = u_x\,dx + u_y\,dy$ and $dv = v_x\,dx + v_y\,dy$, giving 4 equations in the 4 unknowns u_x, u_y, v_x, v_y:

$$\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix}}_{[A]} \begin{bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ du \\ dv \end{bmatrix}, \qquad u_x = \frac{N}{|A|} \ \text{(Cramer's rule)}$$

Characteristic lines are the directions along which the derivatives are *indeterminate*, i.e. $|A| = 0$. Expanding $|A| = 0$ gives

$$A\left(\frac{dy}{dx}\right)^2 + B\,\frac{dy}{dx} + C = 0, \quad A = a_1 c_2 - a_2 c_1, \quad B = -(a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1), \quad C = b_1 d_2 - b_2 d_1$$

(Anderson calls these a, b, c. Capitals are used here so they don't clash with the wave speed c later.) Classification uses the same $D = B^2 - 4AC$.

Eigenvalue method (§3.3): write the system as $\partial W/\partial x + [N]\,\partial W/\partial y = 0$ with $[N] = [K]^{-1}[M]$. Then: all eigenvalues real and distinct → hyperbolic; real but repeated (not enough eigenvectors) → parabolic; complex → elliptic.

0.3 Hyperbolic behaviour: domain of dependence and region of influence (Fig. 3.3)

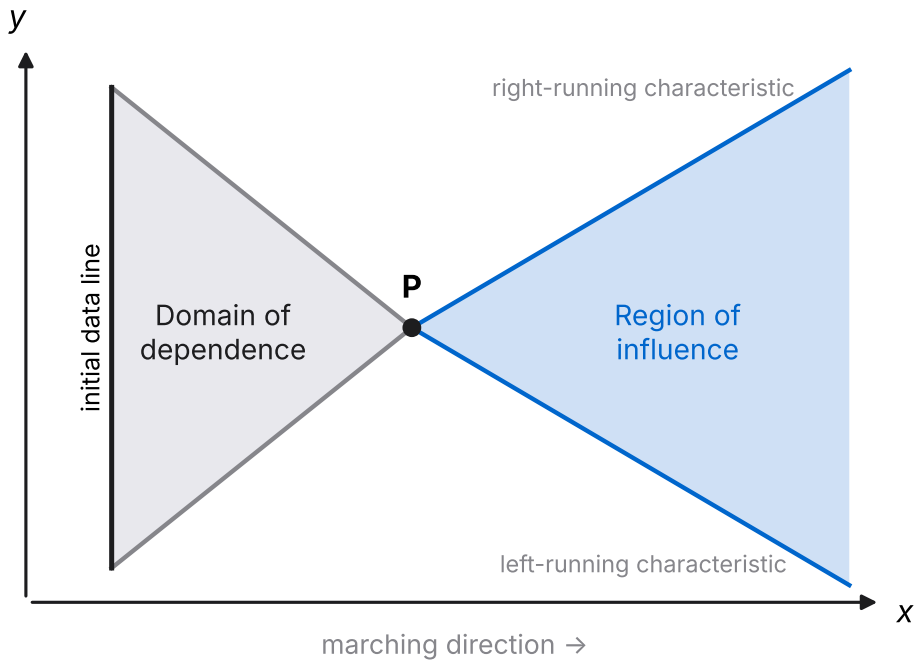


Fig. A (cf. Anderson Fig. 3.3). Through point P run two real characteristics. Information at P influences only the wedge *downstream* between them (region of influence). P itself depends only on the wedge *upstream* (domain of dependence), and in the end on the stretch of initial-data line that wedge cuts off. So you can march the solution from the initial data line.

KEY VOCABULARY

Marching solution: start from known data on an initial line (or plane, or at $t = 0$) and step the solution forward in one direction (x or t), one station at a time. **Characteristic**: a line along which information propagates and across which derivatives can be discontinuous.

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1 · Unsteady Inviscid Flow and Time-Marching (end of §3.4.1)

The **unsteady Euler equations** (inviscid flow) are **hyperbolic with respect to time**, *whether the flow is subsonic or supersonic* (Anderson Fig. 3.7). You can therefore march them forward in time from a known flow field at $t = 0$.

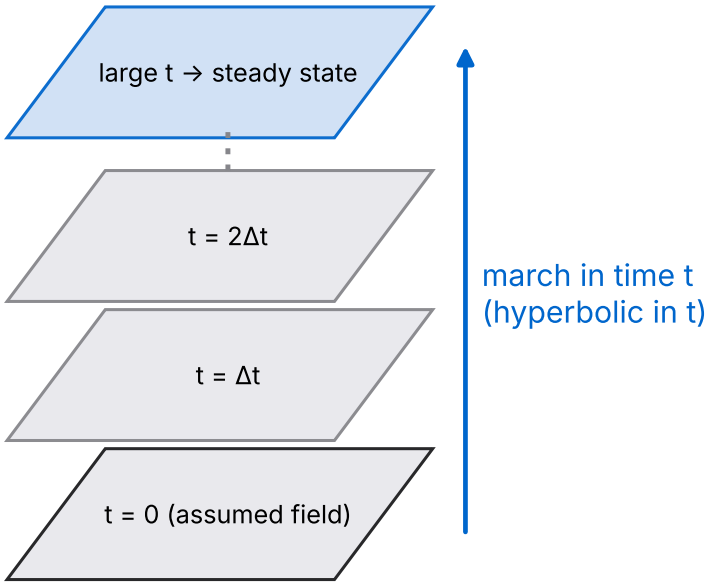


Fig. B (cf. Anderson Fig. 3.7). The unsteady flow field in the xy plane is marched forward in time. If the boundary conditions don't change with time, the solution settles to the steady state at large t.

WHEN DO WE MEET UNSTEADY INVISCID FLOW?

1. **Truly unsteady problems**, where the transient itself is what you want:
   - 1-D **wave motion in a duct**, where you're interested in the transient variations.
   - 2-D unsteady flow over a **flapping or plunging airfoil**.
2. **By far the most common use:** time-marching to get a **steady flow** in the limit of large times, *provided the boundary conditions are time-invariant*. Here time marching is just **"a means to the end"**, and the end is a steady-state flow field.

WHY GO TO THE TROUBLE OF ADDING TIME? (EXAM FAVOURITE)

At first glance it looks inefficient to compute a steady flow by bringing in time as an extra independent variable. But sometimes this is the **only way to get a well-posed problem**, and so the only way to get the steady solution computationally. The classic case is the **supersonic blunt body problem** (Sec. 1.5; see §4 below).

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2 · Parabolic Equations (§3.4.2)

2.1 Two dimensions (Fig. 3.8)

- A parabolic equation in x, y has **only one characteristic direction** through any point P.

- Initial conditions are given on the line **ac**. Boundary conditions are known along the curves **ab** and **cd**.
- The characteristic is a **vertical line through P** ($x = \text{const}$).
- Information at P influences the **entire region on one side** (downstream) of that vertical characteristic, **contained between the two boundaries**. "If we jab P with a needle, the effect is felt throughout the shaded region."
- Like hyperbolic equations, parabolic equations lend themselves to **marching solutions**. Starting from ac, march in the general x direction between the boundaries ab and cd.

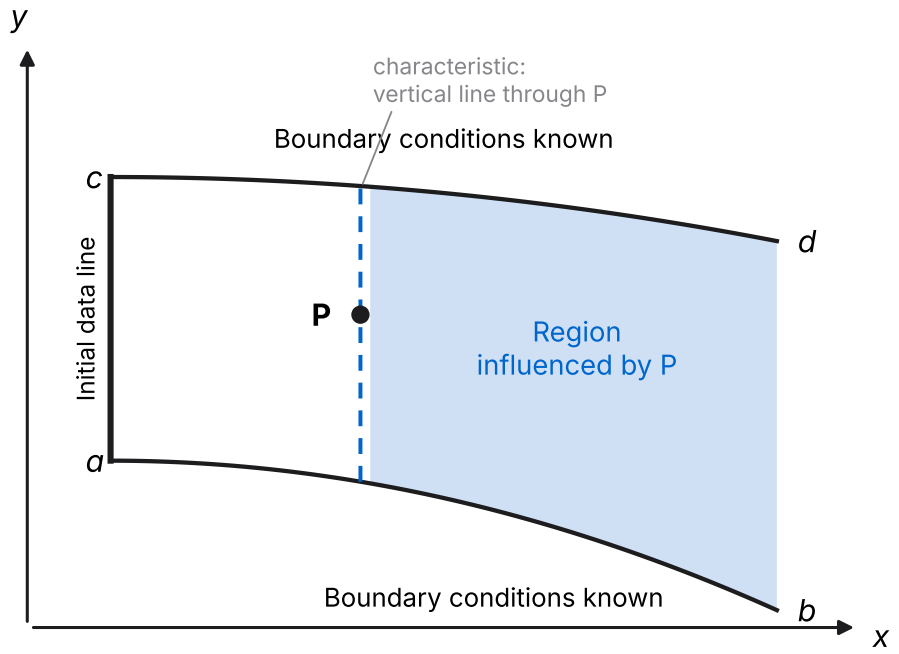


Fig. 3.8. Domain and boundaries for solving parabolic equations in two dimensions. Shaded: the region influenced by P.

2.2 Three dimensions (Fig. 3.9)

- Three independent variables x, y, z. **Initial conditions** are given over the **area abcd in the yz plane**.
- **Boundary conditions** are given on the four surfaces **abgh, cdef, ahed, bgfc**, which extend in the general x direction away from the perimeter of the initial data surface.
- Information at P influences the **whole 3-D region to the right of P**, inside the boundary surfaces (cross-hatched).
- Starting from the initial data plane abcd, the solution is **marched in x**, just like a hyperbolic equation.

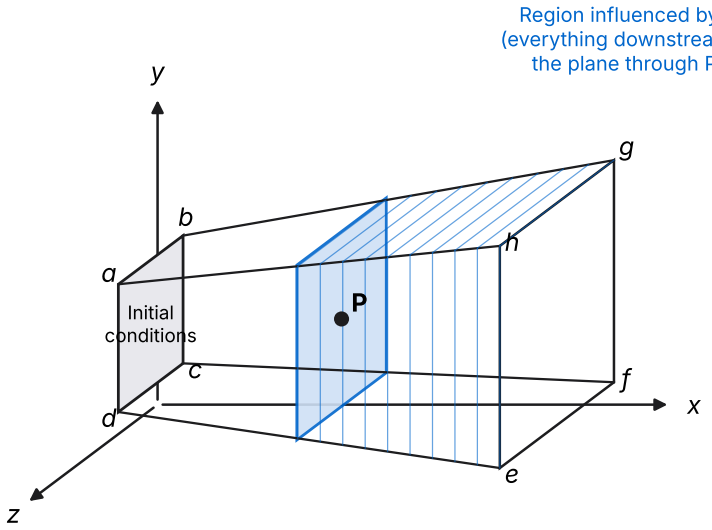


Fig. 3.9. Domain and boundaries for solving parabolic equations in three dimensions.

WHICH FLOWS ARE PARABOLIC?

The full Navier–Stokes equations show **mixed** mathematical behaviour. But many **approximate forms** derived from them are parabolic. So when we ask "which flows are parabolic?", we are really asking **which approximate flow-field *models* are parabolic**. The same idea was used for hyperbolic flows: the Euler equations are themselves a simplified version of NS for inviscid flow. Three parabolic models follow: **(a) steady boundary layers, (b) PNS, (c) unsteady heat conduction.**

2.3 (a) Steady boundary-layer flows (Fig. 3.10)

- **Prandtl's idea (1904, Third Congress of Mathematicians, Heidelberg, Germany):** divide the flow into (1) a **thin layer next to the surface** that contains *all* the viscous effects (the **boundary layer**), and (2) an **inviscid flow outside it**. One of the most profound developments in fluid dynamics.
- Assumptions: the boundary layer is **thin**, and the **Reynolds number based on body length L is large**:

$$\mathrm{Re} = \frac{\rho_\infty V_\infty L}{\mu_\infty} \gg 1$$

  Under these assumptions NS reduces to the **boundary-layer equations**, which are **parabolic**.
- Because they're parabolic, they are solved by **marching downstream in the s direction**, where **s = distance along the body surface measured from the nose**.

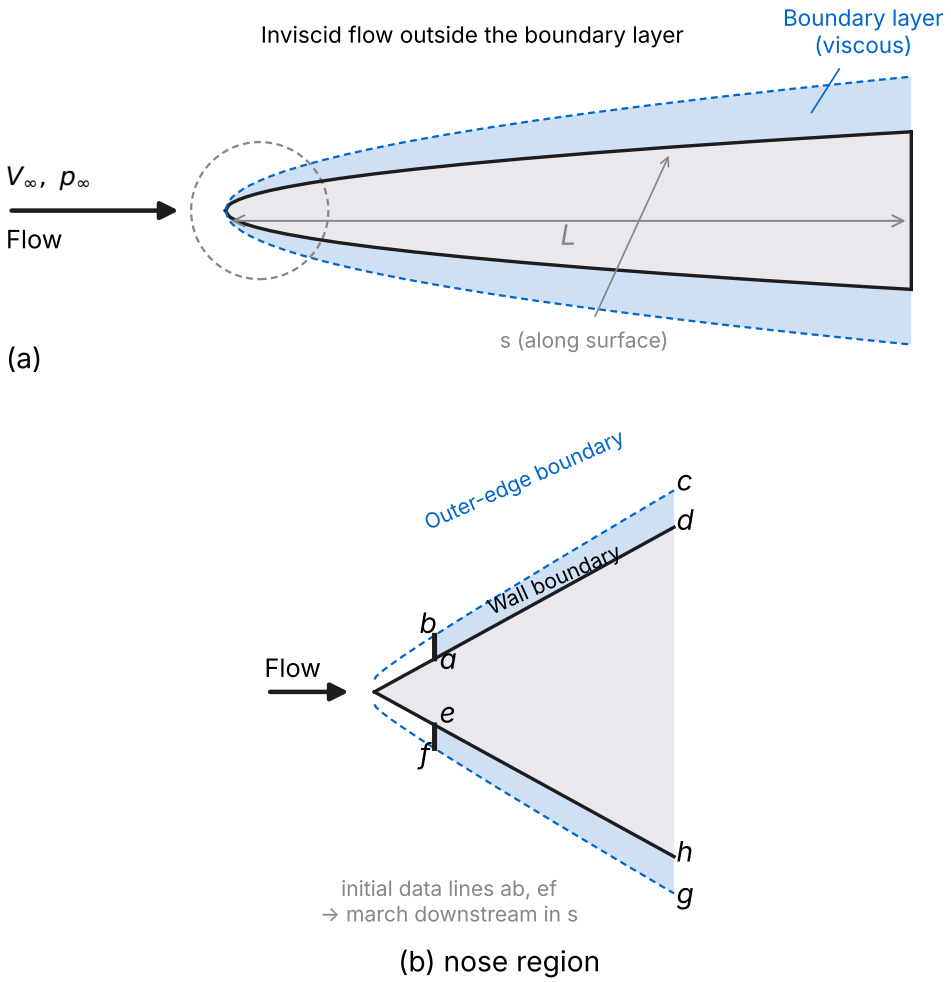


Fig. 3.10. (a) Thin viscous boundary layer (shaded) on a body of length L; inviscid flow outside it. (b) Nose detail. Initial data lines ab and ef run across the layer; ad and eh are the wall boundary; bc and fg are the outer-edge boundary.

Item in Fig. 3.10b	What is specified
ab, ef (initial data lines across the BL)	Initial conditions, taken from a separate **self-similar solution**: a **flat wedge** if the body is 2-D, or a **sharp cone** if it is axisymmetric.
ad, eh (wall)	**No-slip** boundary conditions (Sec. 2.9).
bc, fg (outer edge)	The (usually) known **inviscid flow** conditions. A tenet of **first-order BL theory**: the conditions at the outer edge equal those found *on the body surface* by a purely inviscid solution.

> **SUMMARY SENTENCE TO MEMORISE**
> Because the boundary-layer equations are parabolic, they are solved by **marching downstream in s from an initial data line**, while the **wall and outer-edge boundary conditions** are satisfied at each s station.

2.4 (b) "Parabolized" viscous flows: PNS equations (Fig. 3.11)

- What if the viscous layer is **not thin**? Example: **supersonic flow over a pointed-nose body at low Reynolds number**. Viscous effects reach far from the surface, and the whole region between the shock and the body may be **totally viscous** (a "totally viscous shock layer"). Boundary-layer theory is **not valid** here.
- If the flow has **no regions of localised, reversed or separated flow in the streamwise direction**, another simplified form of NS can work:
  - **Neglect** all viscous terms with **streamwise (x) derivatives**, such as $\frac{\partial}{\partial x}\left(\lambda \nabla \cdot \mathbf{V} + 2\mu \frac{\partial u}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial x}\right)$, $\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)$

- and assume the flow is **steady**.

The result is the **Parabolized Navier–Stokes (PNS) equations**, which behave **parabolically**.

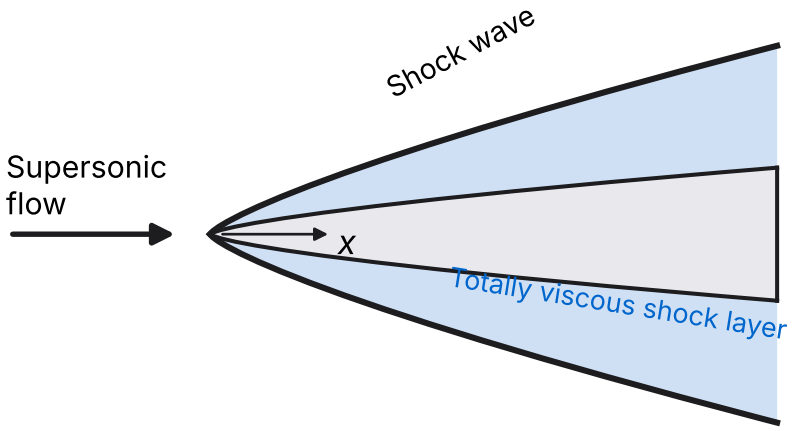


Fig. 3.11. A totally viscous shock layer on a pointed body in supersonic flow. This well-behaved, fully viscous flow is ideal for PNS, and the accuracy is usually quite acceptable.

PNS ADVANTAGES

1. **Simpler**: fewer terms than full NS.
2. Can be solved by **downstream marching**. This is such a compelling advantage that PNS is in widespread use.

PNS LIMITATION

The neglected streamwise viscous terms are the physical mechanism that feeds information **upstream**. So PNS **cannot** handle flows with **streamwise separation**. This is a severe limitation for some applications.

2.5 (c) Unsteady thermal conduction (Eqs 3.25–3.28, Fig. 3.12)

A **stationary fluid** ($\mathbf{V} = 0$) where heat moves by conduction and temperature gradients change with time, for example because of a time-varying wall temperature. Strictly this isn't a "flow".

DERIVATION STEP BY STEP

Step 1. Start from the energy equation in terms of internal energy (Anderson Eq. 2.73): $\rho \dfrac{De}{Dt} = \rho \dot{q} + \nabla \cdot (k \nabla T) - p \nabla \cdot \mathbf{V} + (\text{viscous dissipation})$. With $\mathbf{V} = 0$: $De/Dt = \partial e/\partial t$ (no convection), $p \nabla \cdot \mathbf{V} = 0$, and viscous dissipation $= 0$. So:

$$\rho \frac{\partial e}{\partial t} = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) \tag{3.25}$$

Step 2. No volumetric heat addition ($\dot{q} = 0$), and the state relation $e = c_v T$:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_v}\left[\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right] \tag{3.26}$$

This governs how T varies in time and space. It is **parabolic with respect to time**, which allows a time-marching solution.

Step 3. With constant k, this becomes the well-known **heat conduction equation**, again parabolic:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T, \qquad \alpha = \frac{k}{\rho c_v} \ (\text{thermal diffusivity}) \tag{3.27}$$

PHYSICAL MEANING OF A

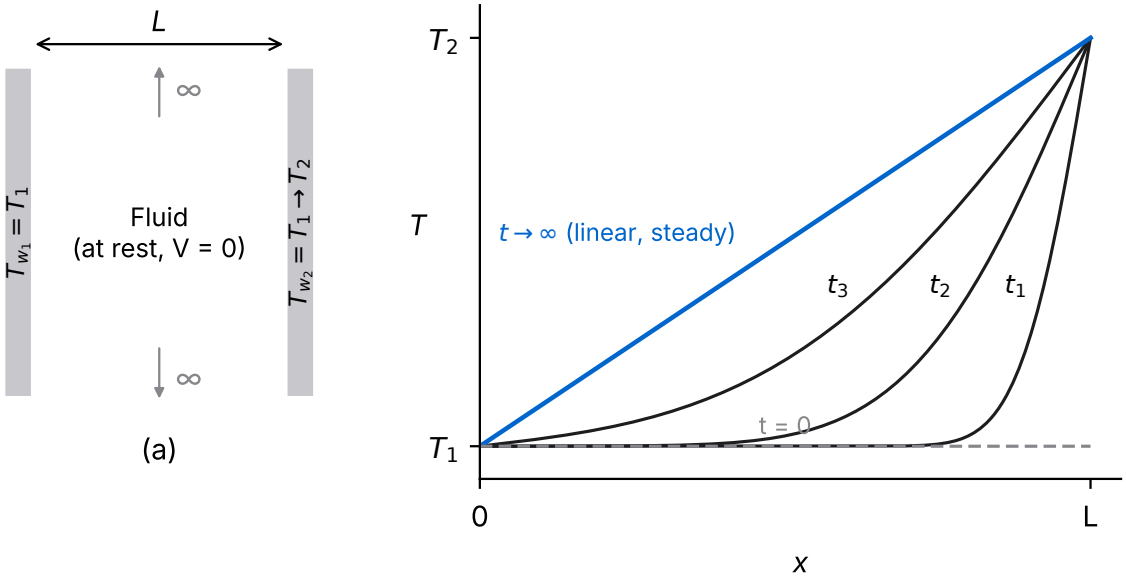
α measures a fluid element's ability to **conduct** energy by thermal conduction, compared with its ability to **retain** that energy (its capacity to absorb heat). Units: m²/s. A large α means temperature changes spread quickly.

EXAMPLE: FLUID BETWEEN TWO PARALLEL WALLS (FIG. 3.12)

- A semi-infinite fluid between two parallel walls a distance L apart, with constant ρ, c_v and k.
- Initially the fluid is at uniform $T = T_1$, in equilibrium with both walls: $T_{w_1} = T_{w_2} = T_1$.
- At $t = 0$ the **right wall** is **impulsively raised** to $T_{w_2} = T_2$. The left wall stays at $T_{w_1} = T_1$.
- The transient is governed by the 1-D form:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.28}$$

- The profiles go from flat at $t = 0$, through curves at $t_2 > t_1 > 0$, to the **linear steady-state distribution as** $t \to \infty$.



(b) $t_3 > t_2 > t_1 > 0$ (exact solution of Eq. 3.28)

Fig. 3.12. (a) Setup. (b) Transient temperature distributions after an impulsive rise of T_{w_2} from T_1 to T_2 at $t = 0$. Curves are plotted from the exact solution below.

EXTRA INSIGHT (NOT IN THE TEXT, BUT IT HELPS YOU UNDERSTAND FIG. 3.12)

Separation of variables gives the exact solution

$$\frac{T - T_1}{T_2 - T_1} = \frac{x}{L} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{L} e^{-n^2 \pi^2 \alpha t / L^2}$$

At $t = 0$ the series cancels x/L exactly, so $T = T_1$. As $t \to \infty$ every exponential dies out, leaving the **straight line** x/L. The characteristic time is $t \sim L^2/\alpha$. Notice that the effect of the wall change is felt **everywhere** in $0 < x < L$ instantly, but **only for later times**: this is parabolic behaviour in t.

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3 · Elliptic Equations (§3.4.3)

- The characteristic curves are **imaginary**, so the **method of characteristics is useless** for elliptic equations.
- There are **no limited regions of influence or domains of dependence**. Information propagates **everywhere, in all directions**.

- The domain is a **closed** rectangle abcd, unlike the **open** domains of hyperbolic (Fig. 3.3) and parabolic (Fig. 3.8) problems.
- Jab P with a needle and the disturbance is felt **everywhere** in the domain. Since P influences every point, the solution at P is in turn influenced by the **entire closed boundary abcd**.
- So the solution at P must be found **simultaneously** with the solution at all other points. This is in stark contrast to marching.
- Elliptic problems are therefore called **"jury problems"**: the solution inside depends on the **total boundary**, so BCs must be applied over the **entire boundary** abcd.

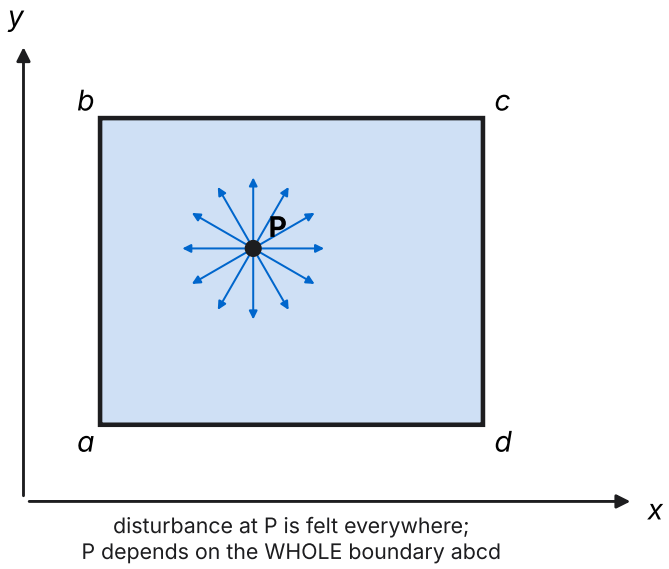


Fig. 3.13. Domain and boundaries for solving elliptic equations in two dimensions.

3.1 Types of boundary condition (on the whole boundary)

Name	What is specified on the boundary	Example
Dirichlet	The **dependent variables** themselves (u, v)	$u = u_b(s)$ on abcd
Neumann	**Derivatives** of the dependent variables	$\partial u/\partial x$ given; for a wall, $\partial \phi/\partial n = 0$
Mixed	A mix of Dirichlet and Neumann	Dirichlet on part of the boundary, Neumann on the rest

The book misprints "Newmann". The correct spelling is **Neumann**. Memory aid: **D**irichlet = **D**ata value; **N**eumann = **N**ormal derivative.

3.2 Flows governed by elliptic equations

(I) STEADY, SUBSONIC, INVISCID FLOW

- The key word is **"subsonic"**. Disturbances travel at the speed of sound or faster, so in subsonic flow they can work their way **upstream** as far as they like.
- In theory, a finite disturbance in an inviscid subsonic flow (no dissipation by friction, thermal conduction or mass diffusion) **propagates to infinity in all directions**.
- **Fig. 3.14** (smoke photograph): streamlines **ahead of** the airfoil are already deflected **upward**, and those **behind** it are deflected **downward**. The airfoil is felt throughout the field, **including far upstream**. This is the physical picture of elliptic behaviour.
- Mathematically, the **steady Euler equations are elliptic when the local Mach number is below 1** (Example 3.1), and hyperbolic when it is above 1.

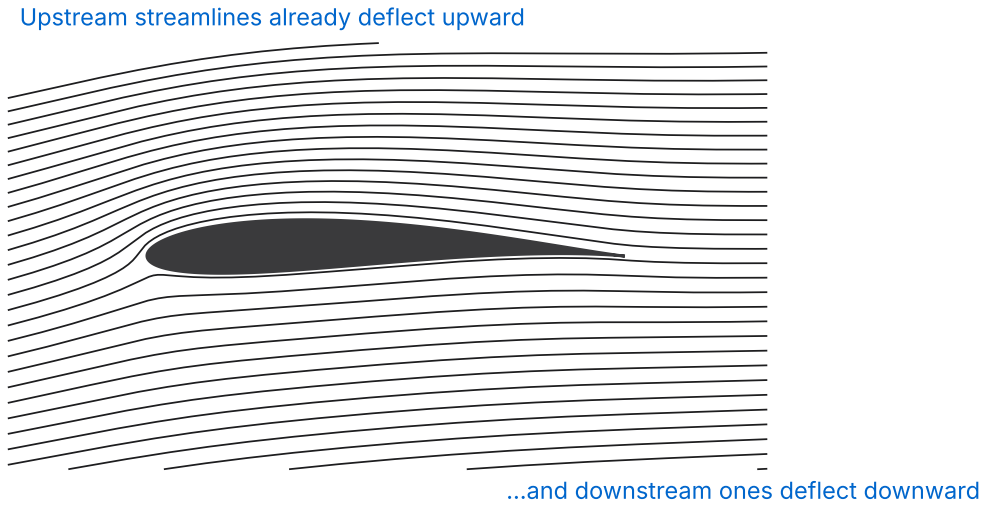


Fig. 3.14 (redrawn). Low-speed subsonic flow over an airfoil, computed here from potential-flow theory. Note the upwash ahead and downwash behind: the disturbance is felt everywhere. (Original photograph courtesy of Hikaru Ito, Meiji University, Japan.)

(II) INCOMPRESSIBLE, INVISCID FLOW

- Incompressible flow is the **limiting case of subsonic flow** as the Mach number goes to zero.
- $M = V/a$. In theory, incompressible flow has zero compressibility, so the **speed of sound is infinite**: $a \to \infty \Rightarrow M = 0$ even though V is finite.
- So incompressible inviscid flow is governed by elliptic equations. It is the **"queen" of elliptic behaviour**: everything said about subsonic flow applies, with a **stronger** effect. For irrotational flow it reduces to **Laplace's equation** $\nabla^2\phi = 0$ (Problem 3.3).

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4 · Some Comments: The Supersonic Blunt Body Problem Revisited (§3.4.4)

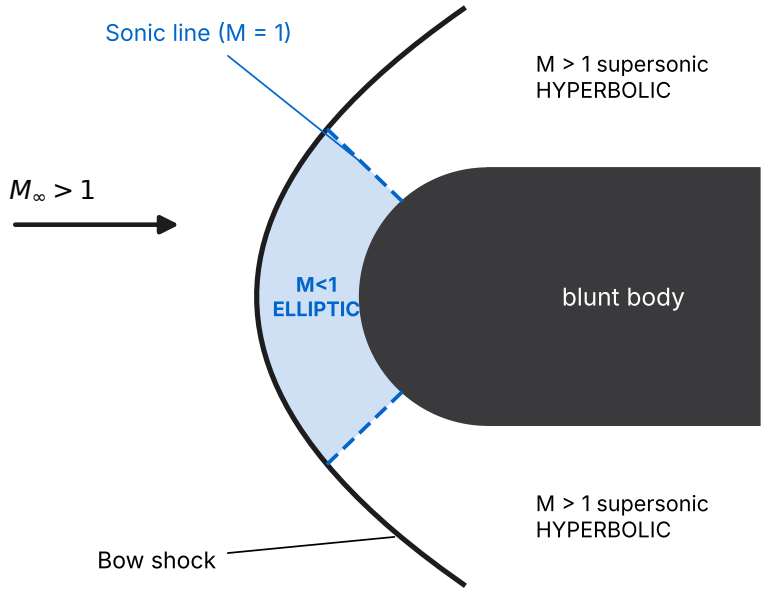


Fig. C (cf. Anderson Fig. 1.30). Supersonic flow over a blunt body. Behind the strong, nearly normal part of the bow shock there is a locally **subsonic** (elliptic) region near the nose. Away from it the flow is locally **supersonic** (hyperbolic). The two are separated by the **sonic line**.

- This is one of the most important problems in modern high-speed aerodynamics: the inviscid flow over a supersonic or hypersonic blunt body.
- **The difficulty:** the steady flow is **mixed**. It is subsonic (**elliptic**) near the nose and supersonic (**hyperbolic**) further downstream. The steady Euler equations change their mathematical nature abruptly across the sonic line, which **virtually rules out** any practical steady-flow method that treats both regions uniformly. Up to the mid-1960s, no single method worked in both regions.
- **The breakthrough (mid-1960s):** the **unsteady** inviscid (Euler) equations are **hyperbolic with respect to time** whether the flow is locally subsonic or supersonic (Fig. 3.7). So:
  1. Start from **rather arbitrary initial conditions** at $t = 0$.
  2. **March forward in time** with the unsteady 2-D Euler equations.
  3. At large times the time derivatives approach zero, and the **steady state** you reach is the solution for the **entire field**, both subsonic and supersonic parts, from **one uniform method**.
- This is the **time-dependent technique**. Its practical numerical implementation by **Moretti and Abbett (1966)** was the major scientific breakthrough for the blunt body problem.
- Time looks like "extra baggage" but it is essential. **Without time as an independent variable the problem cannot be solved.** With time, the Euler equations are hyperbolic in t, so marching is straightforward and the steady result appears at large times. The marching is just **a means to an end**.

> **LESSON (ANDERSON'S ITALICS)**
> This is a classic example of **why it matters to understand the mathematical behaviour of different types of PDEs**. Using that understanding intelligently produced a practical solution where none existed before.

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5 · The Navier–Stokes Equations Are Mixed

- Time-marching to a steady state is used widely across modern CFD. It is in no way unique to the blunt body problem.
- The **full unsteady Navier–Stokes equations** don't fit a single category. They have **both parabolic and elliptic** behaviour:
  - **Parabolic** through the **time derivatives** of velocity and internal energy, just as the heat equation (3.27) is parabolic through $\partial T/\partial t$.
  - **Partly elliptic** through the **viscous terms**, which feed information **upstream**.
- Despite this mixed nature, a **time-marching solution is well-posed**, and most existing solvers for the full compressible NS equations use time-marching.

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6 · Well-Posed Problems (§3.5)

It is easy to attempt a solution with **incorrect or insufficient** boundary or initial conditions. Analytical or numerical, such an **ill-posed** problem usually gives **spurious results at best and no solution at worst**.

> **DEFINITION (MEMORISE WORD FOR WORD)**
> If the solution to a PDE **(1) exists**, **(2) is unique**, and **(3) depends continuously on the initial and boundary conditions**, then the problem is **well-posed**. (These are often called Hadamard's conditions.)

- In CFD, **establish that your problem is well-posed before attempting a numerical solution.**
- **Ill-posed example:** the blunt body treated as a *steady* flow. Any attempt at a uniformly valid method for both the subsonic and supersonic regions was ill-posed.
- **Became well-posed:** the same problem set up with the **unsteady** Euler equations and time-marched to steady state from essentially arbitrary initial conditions at $t = 0$.

- "Continuous dependence" means a small change in the IC/BC gives only a small change in the solution. Classic ill-posed examples (extra): the heat equation run **backward** in time, or Laplace's equation given as an initial-value (Cauchy) problem. Both blow up small errors without bound.

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7 · Summary (§3.6) and Master Comparison

- There are **two standard methods** for finding a PDE's mathematical behaviour: the **Cramer's rule approach (§3.2)** and the **eigenvalue method (§3.3)**.
- Many equations are clearly hyperbolic, parabolic or elliptic. Others, like the **unsteady Navier–Stokes equations, are mixed**.
- **Hyperbolic and parabolic** equations lend themselves to **marching** from a known initial data line or plane. **Elliptic** equations don't: the variables at a point must be solved **simultaneously** with all other points.
- The different **mathematical** behaviour directly reflects the different **physical** behaviour of the flows.
- This ends Part I of the book (basic ideas and equations). Part II covers the numerical aspects of CFD.

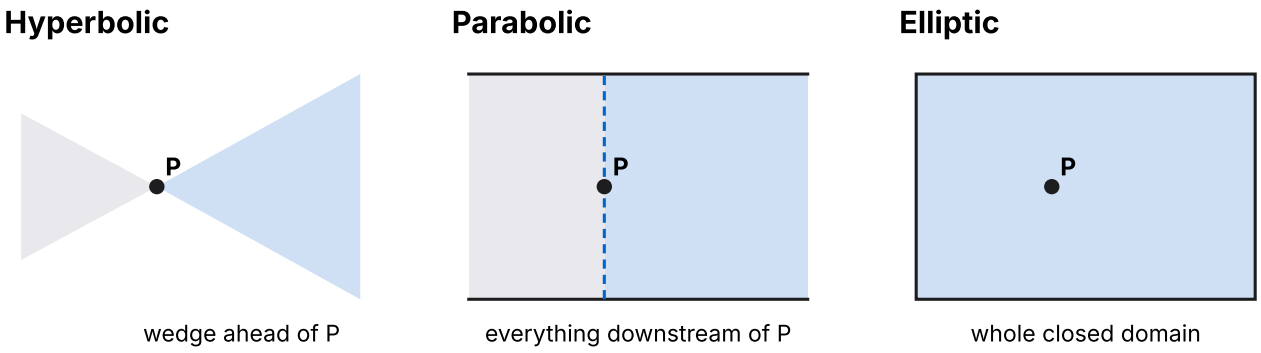


Fig. D. The region influenced by a disturbance at P for each class.

	Hyperbolic	Parabolic	Elliptic
$B^2 - 4AC$	> 0	= 0	< 0
Characteristics	2 real, distinct	1 real	imaginary
Region of influence of P	wedge between the characteristics, downstream	whole region downstream of the characteristic through P, between the boundaries	the entire domain
Domain	open	open	closed
Data needed	initial data line + BCs	initial data line + BCs on the side boundaries	BCs on the *entire* boundary (Dirichlet / Neumann / mixed)
Solution method	marching	marching	simultaneous ("jury" problem)
Model equation	$u_{tt} = c^2 u_{xx}$, $u_t + c u_x = 0$	$T_t = \alpha T_{xx}$	$\phi_{xx} + \phi_{yy} = 0$
Flow examples	steady supersonic inviscid; unsteady inviscid (any M)	steady BL; PNS; unsteady conduction	steady subsonic inviscid; incompressible inviscid

8 · Problems 3.1–3.5, Fully Solved

Problem 3.1: Compatibility equation along the characteristics

By expanding the determinant in Eq. (3.14), obtain the compatibility equation which holds along the characteristic lines.

Idea. Along a characteristic, the denominator $|A| = 0$. But the derivative $u_x = N/|A|$ must stay finite (the solution is physical), so the **numerator must also vanish**: $N = 0$. Eq. (3.14) is that numerator determinant, made by replacing the first column of $[A]$ with the right-hand side. Expanding $N = 0$ gives a relation between du and dv that holds **only along the characteristic lines**: the **compatibility equation**.

$$N = \begin{vmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ du & dy & 0 & 0 \\ dv & 0 & dx & dy \end{vmatrix} = 0 \tag{3.14}$$

Step 1: expand along row 3, which has only two non-zero entries, du (column 1, sign $+$) and dy (column 2, sign $-$):

$$N = du \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ 0 & dx & dy \end{vmatrix} - dy \begin{vmatrix} f_1 & c_1 & d_1 \\ f_2 & c_2 & d_2 \\ dv & dx & dy \end{vmatrix}$$

Step 2: first 3×3 (expand along its first column): $b_1(c_2dy - d_2dx) - b_2(c_1dy - d_1dx) = (b_1c_2 - b_2c_1)\,dy - (b_1d_2 - b_2d_1)\,dx$

Step 3: second 3×3 (expand along its first column): $f_1(c_2dy - d_2dx) - f_2(c_1dy - d_1dx) + dv\,(c_1d_2 - c_2d_1)$

Step 4: combine and set N = 0:

$$\big[(b_1c_2 - b_2c_1)\,dy - (b_1d_2 - b_2d_1)\,dx\big]du + (c_2d_1 - c_1d_2)\,dy\,dv = (f_1c_2 - f_2c_1)(dy)^2 + (f_2d_1 - f_1d_2)\,dx\,dy$$

Step 5: divide by dx and write $\lambda = dy/dx$, the characteristic slope found from $A\lambda^2 + B\lambda + C = 0$:

> **ANSWER: COMPATIBILITY EQUATION**
>
> $$\big[(b_1c_2 - b_2c_1)\lambda - (b_1d_2 - b_2d_1)\big]\,du + (c_2d_1 - c_1d_2)\,\lambda\,dv = \big[(f_1c_2 - f_2c_1)\lambda + (f_2d_1 - f_1d_2)\big]\lambda\,dx$$
>
> It holds **only along** the characteristics $dy/dx = \lambda$. It turns the PDEs into ODEs along those lines, which is the basis of the **method of characteristics**. (Checked by symbolic algebra.)

CHECK WITH EXAMPLE 3.1 (LINEARISED STEADY FLOW)

$(1 - M^2)u_x + v_y = 0$ and $u_y - v_x = 0$ give $a_1 = 1 - M^2,\ d_1 = 1,\ b_2 = 1,\ c_2 = -1$, all others zero, $f_1 = f_2 = 0$.
Characteristics: $A = a_1c_2 - a_2c_1 = M^2 - 1,\ B = 0,\ C = b_1d_2 - b_2d_1 = -1$, so $D = 4(M^2 - 1)$, and $\lambda = \pm\dfrac{1}{\sqrt{M^2 - 1}}$ (the **Mach lines**; hyperbolic for M > 1).
Compatibility: $b_1c_2 - b_2c_1 = 0,\ b_1d_2 - b_2d_1 = -1,\ c_2d_1 - c_1d_2 = -1$, so $du - \lambda\,dv = 0$, i.e.

$$du = \pm\frac{dv}{\sqrt{M^2 - 1}} \quad \text{along} \quad \frac{dy}{dx} = \pm\frac{1}{\sqrt{M^2 - 1}}$$

Sanity check: a simple wave $u = F(x - \beta y)$, $v = -\beta F$ with $\beta = \sqrt{M^2 - 1}$ satisfies both PDEs, and along $\lambda = -1/\beta$ we get $du = dF = -\frac{1}{\beta}(-\beta\,dF)$ ✓.

Problem 3.2: Prove the heat conduction equation is parabolic

Prove that $\dfrac{\partial T}{\partial t} = \alpha\dfrac{\partial^2 T}{\partial x^2}$ *is a parabolic equation.*

METHOD 1: SECOND-ORDER DISCRIMINANT

Take the independent variables as (x, t) in place of (x, y) and rewrite: $\alpha\, T_{xx} + 0 \cdot T_{xt} + 0 \cdot T_{tt} - T_t = 0$.
So $A = \alpha,\ B = 0,\ C = 0$ (the T_t term is lower order and doesn't affect the type).

$$D = B^2 - 4AC = 0 - 4(\alpha)(0) = 0 \ \Rightarrow\ \textbf{parabolic}$$

Characteristics: $A(dt)^2 - B\,dx\,dt + C(dx)^2 = \alpha(dt)^2 = 0 \Rightarrow dt = 0$. There is **one real characteristic family,** $t =$ **const**: horizontal lines in the x–t plane. So information at time t affects all x at once, but only for **later** times, and the solution can be marched in t.

METHOD 2: CRAMER'S RULE (ANDERSON'S METHOD)

Let $u = T$ and $v = q \equiv \partial T/\partial x$, and use (x, t) for Anderson's (x, y):
Eq. 1: $T_t - \alpha\, q_x = 0 \ \Rightarrow\ a_1 = 0,\ b_1 = 1,\ c_1 = -\alpha,\ d_1 = 0,\ f_1 = 0$
Eq. 2: $T_x = q \ \Rightarrow\ a_2 = 1,\ b_2 = 0,\ c_2 = 0,\ d_2 = 0,\ f_2 = q$
$A = a_1c_2 - a_2c_1 = 0 - (1)(-\alpha) = \alpha$
$B = -(a_1d_2 - a_2d_1 + b_1c_2 - b_2c_1) = -(0 - 0 + 0 - 0) = 0$
$C = b_1d_2 - b_2d_1 = 0$
$D = B^2 - 4AC = 0 \ \Rightarrow$ **parabolic** ■. And $\alpha\,(dt/dx)^2 = 0 \Rightarrow$ characteristics $t =$ const, as before.

Problem 3.3: Show Laplace's equation is elliptic

Show that $\phi_{xx} + \phi_{yy} = 0$ is an elliptic equation.

METHOD 1: DISCRIMINANT

$A = 1,\ B = 0,\ C = 1$, so $D = 0 - 4(1)(1) = -4 < 0\ \Rightarrow$ **elliptic**. Characteristics: $(dy/dx)^2 + 1 = 0 \Rightarrow dy/dx = \pm i$, which is **imaginary**, so there are no real characteristic lines.

METHOD 2: CRAMER'S RULE

Let $u = \phi_x,\ v = \phi_y$. Then Laplace gives $u_x + v_y = 0$, and $\phi_{xy} = \phi_{yx}$ gives $u_y - v_x = 0$.
$a_1 = 1,\ b_1 = 0,\ c_1 = 0,\ d_1 = 1; \quad a_2 = 0,\ b_2 = 1,\ c_2 = -1,\ d_2 = 0$
$A = a_1c_2 - a_2c_1 = -1, \quad B = -(0 - 0 + 0 - 0) = 0, \quad C = b_1d_2 - b_2d_1 = -1$
$D = 0 - 4(-1)(-1) = -4 < 0\ \Rightarrow$ **elliptic** ■. Note this is just Example 3.1 with $M = 0$: incompressible flow is the "queen" of elliptic behaviour.

METHOD 3: EIGENVALUES

$\dfrac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dfrac{\partial}{\partial y}\begin{bmatrix} u \\ v \end{bmatrix} = 0$. Then $\det(N - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$, which is complex, so elliptic.

Problem 3.4: Show the second-order wave equation is hyperbolic

Show that $u_{tt} = c^2 u_{xx}$ is hyperbolic.

METHOD 1: DISCRIMINANT

Write it as $c^2 u_{xx} + 0 \cdot u_{xt} - u_{tt} = 0$ in (x, t): $A = c^2,\ B = 0,\ C = -1$.

$$D = 0 - 4(c^2)(-1) = 4c^2 > 0 \ \Rightarrow\ \textbf{hyperbolic}$$

Characteristics: $c^2\lambda^2 - 1 = 0$ with $\lambda = dt/dx$, so $\lambda = \pm\frac{1}{c}$, i.e. $\dfrac{dx}{dt} = \pm c$. These are **two real families** $x - ct =$ **const and** $x + ct =$ **const**: right- and left-running waves at speed c.

METHOD 2: CRAMER'S RULE

Let $p = u_t,\ q = u_x$. Then $p_t - c^2 q_x = 0$ and $q_t - p_x = 0$ (since $u_{xt} = u_{tx}$). With Anderson's $u \to p,\ v \to q,\ y \to t$:
Eq. 1: $a_1 = 0,\ b_1 = 1,\ c_1 = -c^2,\ d_1 = 0$ Eq. 2: $a_2 = -1,\ b_2 = 0,\ c_2 = 0,\ d_2 = 1$
$A = a_1c_2 - a_2c_1 = 0 - (-1)(-c^2) = -c^2, \quad B = -(0 - 0 + 0 - 0) = 0, \quad C = b_1d_2 - b_2d_1 = 1$
$D = 0 - 4(-c^2)(1) = 4c^2 > 0\ \Rightarrow$ **hyperbolic** ■, with $-c^2\lambda^2 + 1 = 0 \Rightarrow dx/dt = \pm c$.

BONUS: COMPATIBILITY (USING PROBLEM 3.1)

$b_1c_2 - b_2c_1 = 0,\ b_1d_2 - b_2d_1 = 1,\ c_2d_1 - c_1d_2 = c^2$, so $-dp + c^2\lambda\, dq = 0$, giving $dp = \pm c\, dq$.
Along $dx/dt = +c$: $p - cq = u_t - cu_x =$ const. Along $dx/dt = -c$: $u_t + cu_x =$ const. These are the **Riemann invariants**, consistent with d'Alembert's solution $u = F(x - ct) + G(x + ct)$.
Eigenvalues: $\partial_t[p,q]^T + \begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix} \partial_x[p,q]^T = 0$ gives $\lambda^2 = c^2$, so $\lambda = \pm c$: real and distinct, hence hyperbolic. The eigenvalues *are* the wave speeds.

Problem 3.5: Show the first-order wave equation is hyperbolic

Show that $u_t + c\,u_x = 0$ is hyperbolic.

METHOD 1: CRAMER'S RULE / CHARACTERISTICS

Unknowns u_t, u_x. Equations: $u_t + c\,u_x = 0$ and $du = u_t\,dt + u_x\,dx$:

$$\begin{bmatrix} 1 & c \\ dt & dx \end{bmatrix} \begin{bmatrix} u_t \\ u_x \end{bmatrix} = \begin{bmatrix} 0 \\ du \end{bmatrix}$$

Characteristic: $|A| = dx - c\,dt = 0 \Rightarrow \dfrac{dx}{dt} = c$. This is **real** (lines $x - ct =$ const). A first-order equation has one characteristic, and since it is real the equation is **hyperbolic ■**.
Compatibility: $N = \begin{vmatrix} 0 & c \\ du & dx \end{vmatrix} = -c\,du = 0 \Rightarrow du = 0$. So **u is constant along each characteristic**, and $u(x,t) = F(x - ct)$: the initial profile translates to the right at speed c without changing shape.

METHOD 2: EIGENVALUE

$\partial u/\partial t + [c]\,\partial u/\partial x = 0$. The 1×1 "matrix" has eigenvalue $\lambda = c$, which is **real**, so it is hyperbolic.

METHOD 3: REDUCE TO PROBLEM 3.4

Apply $(\partial_t - c\,\partial_x)$ to $u_t + cu_x = 0$: $(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = u_{tt} - c^2u_{xx} = 0$. This is the second-order wave equation, which is hyperbolic (Problem 3.4).

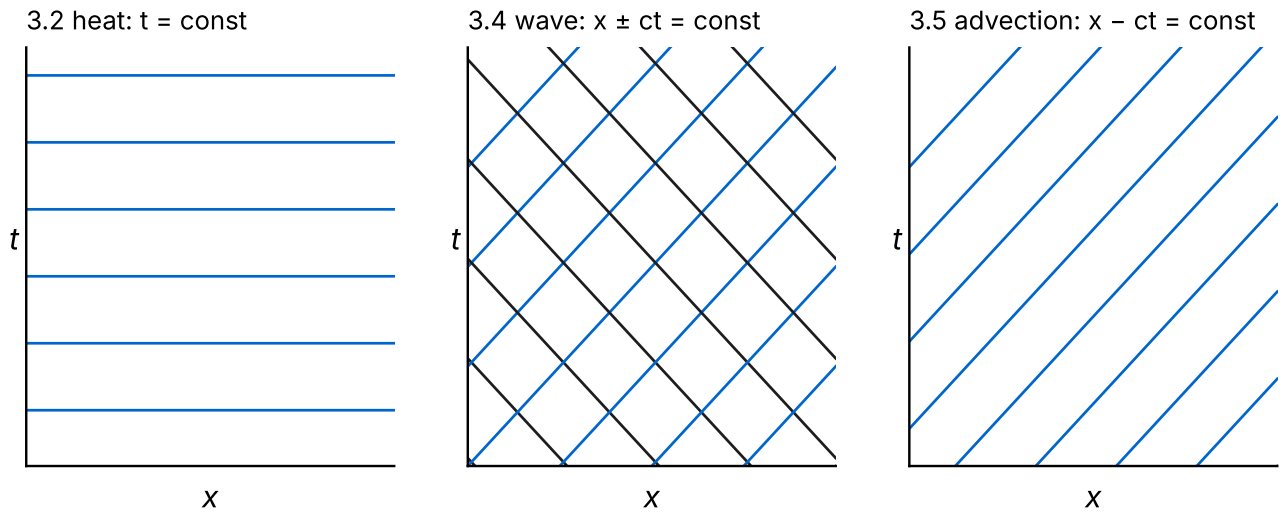


Fig. E. Characteristics in the x–t plane for Problems 3.2, 3.4 and 3.5.

9 · Quiz

Answer everything first, then check the key on the following pages. Suggested time: 30 minutes. Scoring: Sections A–B 1 mark each, C 1 mark each, D 2 marks each, E 3 marks each.

A · Multiple choice (choose one)

1. A parabolic equation in two independent variables has how many characteristic directions through a point?
   (a) none
   (b) one
   (c) two
   (d) infinitely many

2. In Fig. 3.8, the characteristic through P is
   (a) a horizontal line
   (b) a vertical line
   (c) two lines making a wedge
   (d) imaginary

3. The region influenced by P for a parabolic equation is
   (a) the whole closed domain
   (b) a wedge between two characteristics
   (c) the whole region downstream of the characteristic through P, between the boundaries
   (d) only the point P

4. Which is NOT governed by a parabolic equation?
   (a) steady boundary layer
   (b) PNS flow
   (c) unsteady heat conduction
   (d) steady incompressible inviscid flow

5. The boundary-layer concept was presented by Prandtl in
   (a) 1894, Göttingen
   (b) 1904, Heidelberg
   (c) 1924, Berlin
   (d) 1966, Rome

6. The boundary-layer equations are obtained from NS assuming
   (a) thin BL and large Re
   (b) thick BL and small Re
   (c) M < 1
   (d) inviscid flow

7. In a boundary-layer solution, the marching direction is
   (a) normal to the wall
   (b) time
   (c) s, distance along the surface from the nose
   (d) upstream

8. Initial conditions on lines ab and ef in Fig. 3.10b for a 2-D body come from
   (a) a sharp-cone solution
   (b) a flat-wedge self-similar solution
   (c) the PNS equations
   (d) the no-slip condition

9. At the outer edge of the boundary layer (bc, fg) we apply
   (a) no-slip
   (b) zero temperature gradient
   (c) the inviscid flow conditions found on the body surface
   (d) free-stream values always

10. PNS equations are obtained by neglecting viscous terms with derivatives in the
    (a) normal direction
    (b) streamwise direction
    (c) spanwise direction
    (d) time direction

11. The main limitation of PNS is that it cannot handle
    (a) supersonic flow
    (b) shock waves
    (c) streamwise flow separation
    (d) heat transfer

12. In Anderson's notation, the thermal diffusivity is
    (a) $k/\rho c_v$
    (b) $\rho c_v/k$
    (c) μ/ρ
    (d) $k\rho c_v$

13. In Fig. 3.12, the temperature distribution as $t \to \infty$ is
(a) uniform at T_1 (b) uniform at T_2
(c) linear from T_1 to T_2 (d) exponential

14. For elliptic equations, the method of characteristics is useless because characteristics are
(a) parallel (b) imaginary
(c) vertical (d) repeated

15. Elliptic problems are called "jury problems" because
(a) they need a vote (b) the solution depends on the total boundary
(c) they have many solutions (d) they march downstream

16. Specifying $\partial u/\partial x$ on the boundary is a
(a) Dirichlet condition (b) Neumann condition
(c) Cauchy condition (d) initial condition

17. Steady Euler equations are elliptic when
(a) M > 1 (b) M = 1
(c) M < 1 (d) always

18. Incompressible flow has M = 0 because
(a) V = 0 (b) the speed of sound is infinite
(c) density is zero (d) viscosity is zero

19. Unsteady inviscid flow equations are hyperbolic in time
(a) only if subsonic (b) only if supersonic
(c) whether subsonic or supersonic (d) never

20. The practical time-dependent solution of the blunt body problem was by
(a) Prandtl, 1904 (b) Moretti & Abbett, 1966
(c) Navier & Stokes, 1845 (d) Anderson, 1995

21. The full unsteady NS equations are
(a) purely hyperbolic (b) purely elliptic
(c) mixed: parabolic (time terms) and elliptic (viscous terms) (d) purely parabolic

22. Which is NOT part of the definition of a well-posed problem?
(a) solution exists (b) solution is unique
(c) depends continuously on IC/BC (d) solution is linear

23. $u_{xx} + 4u_{xy} + 3u_{yy} = 0$ is
(a) elliptic (b) parabolic
(c) hyperbolic (d) mixed

24. $u_{xx} + 2u_{xy} + u_{yy} = 0$ is
(a) elliptic (b) parabolic
(c) hyperbolic (d) mixed

25. The domain for an elliptic problem is
(a) open (b) closed
(c) semi-infinite in the marching direction (d) a single line

B · True or false

26. Parabolic equations, like hyperbolic ones, lend themselves to marching solutions.

27. The full Navier–Stokes equations are parabolic.

28. PNS equations assume steady flow.

29. In an elliptic problem, the solution at P can be found before the solution at other points.

30. Equation (3.26) is parabolic with respect to time.

31. The time-marching approach is unique to the blunt-body problem.

32. In a theoretically inviscid subsonic flow, a finite disturbance propagates to infinity in all directions.

33. The steady blunt-body problem with a single uniform method for both regions is well-posed.

34. Time-marching solutions of the full compressible NS equations are well-posed.

35. In 3-D parabolic problems, initial conditions are given over a surface in the yz plane.

C · Fill in the blanks

36. When time-marching to steady state, the boundary conditions must be _____.

37. The thin viscous layer next to a surface is called the _____.

38. Specifying the dependent variables themselves on the boundary is a _____ condition.

39. The incompressible inviscid flow is called the "_____" of elliptic behaviour.

40. The line separating the subsonic and supersonic regions in the blunt-body flow is the _____ line.

41. With constant k, Eq. (3.26) becomes $\partial T/\partial t =$ _____.

42. The two standard methods for classifying PDEs are the _____ approach and the _____ method.

43. In Fig. 3.9, boundary conditions are applied on the four surfaces abgh, cdef, _____ and _____.

44. Re based on body length = _____.

45. The PNS-type flow in Fig. 3.11 is called a totally viscous _____.

D · Short answer (2–4 lines each)

46. Why would we ever compute a steady flow by introducing time as an extra variable?

47. Explain physically why steady subsonic inviscid flow is elliptic, using Fig. 3.14.

48. Why can't PNS handle separated flows?

49. Why was the steady blunt-body problem so difficult, and how did the time-dependent technique solve it?

50. Give the definition of a well-posed problem and one example each of ill-posed and well-posed.

51. Contrast the solution procedure for elliptic vs hyperbolic/parabolic equations.

52. State the physical meaning of thermal diffusivity.

53. Where does the parabolic and where does the elliptic behaviour of the unsteady NS equations come from?

E · Problems

54. Classify $(1-M^2)\phi_{xx}+\phi_{yy}=0$ for $M=0.5$ and $M=2$. For $M=2$, find the characteristic slopes and the angle they make with the x axis.

55. Classify the Tricomi equation $y\,u_{xx}+u_{yy}=0$. Which region is like the blunt-body subsonic zone?

56. Air at rest: $k=0.026$ W/(m·K), $\rho=1.2$ kg/m³, $c_v=718$ J/(kg·K). Find α and the characteristic time L^2/α for $L=1$ cm.

57. Show that $T_t=\alpha T_{xx}$ is parabolic and find its characteristics.

58. Show that $u_t+cu_x=0$ is hyperbolic and find what happens to u along the characteristic.

59. Classify $u_{xx}-6u_{xy}+9u_{yy}=0$ and $3u_{xx}+u_{xy}+u_{yy}=0$.

60. Derive Eq. (3.28) from the energy equation, stating every assumption.

9 · Answer Key with Explanations

A · Multiple choice

Q	Ans	Why	Q	Ans	Why
1	b	D = 0 gives one real (repeated) root.	14	b	D < 0, so there are no real characteristics.
2	b	Vertical line through P (x = const).	15	b	The solution depends on the total boundary.
3	c	One side of the characteristic, between ab and cd.	16	b	Derivatives specified → Neumann.
4	d	Incompressible inviscid flow is elliptic ("queen").	17	c	Example 3.1: elliptic for M < 1.
5	b	Third Congress of Mathematicians, Heidelberg, 1904.	18	b	a → ∞, so M = V/a → 0.
6	a	Thin layer, Re = $\rho_\infty V_\infty L/\mu_\infty$ large.	19	c	Fig. 3.7: holds for any M.
7	c	March downstream in s.	20	b	Ref. 12, 1966.
8	b	Flat wedge (2-D); sharp cone (axisymmetric).	21	c	Time derivatives → parabolic; viscous terms → elliptic.
9	c	Tenet of first-order BL theory.	22	d	Exists, unique, continuous dependence only.
10	b	Streamwise viscous derivatives dropped.	23	c	D = 16 − 12 = 4 > 0.
11	c	Dropped terms carry information upstream.	24	b	D = 4 − 4 = 0.
12	a	$\alpha = k/\rho c_v$ (Eq. 3.27).	25	b	Closed (rectangle abcd).
13	c	Steady 1-D conduction gives a straight line.			

B · True or false

26 **T** · 27 **F** (mixed) · 28 **T** · 29 **F** (simultaneous) · 30 **T** · 31 **F** (widely used across CFD) · 32 **T** · 33 **F** (ill-posed) · 34 **T** · 35 **T** (area abcd)

C · Fill in the blanks

36 time-invariant · 37 boundary layer · 38 Dirichlet · 39 queen · 40 sonic · 41 $\alpha\nabla^2 T$ · 42 Cramer's rule; eigenvalue · 43 ahed, bgfc · 44 $\rho_\infty V_\infty L/\mu_\infty$ · 45 shock layer

D · Model answers

46. Sometimes adding time is the only way to get a well-posed problem. The unsteady equations are hyperbolic in time and can be marched, and with time-invariant BCs the solution approaches the steady state at large t. Example: the supersonic blunt body.

47. Disturbances travel at the speed of sound (or faster), which exceeds the flow speed in subsonic flow, so they reach upstream without limit and the airfoil's presence is felt everywhere. Fig. 3.14 shows streamlines deflected upward ahead of the airfoil and downward behind it, i.e. influence in all directions, which is elliptic behaviour.

48. PNS drops the streamwise viscous derivative terms. Those terms are the physical mechanism that feeds information upstream, and separated (reversed) flow needs upstream influence. So PNS is valid only for flows without streamwise separation.

49. The steady flow is mixed: elliptic (subsonic) near the nose and hyperbolic (supersonic) downstream. The equations change type across the sonic line, so no single steady method worked in both regions (the problem was ill-posed). The unsteady Euler equations are hyperbolic in time everywhere, so marching in time from arbitrary initial conditions reaches the steady state for the whole field with one method (Moretti & Abbett, 1966).

50. Well-posed: a solution exists, is unique, and depends continuously on the IC/BC. Ill-posed: the steady blunt-body problem with a uniform method. Well-posed: the same problem as an unsteady time-marching problem.

51. Hyperbolic and parabolic: march from an initial data line or plane, one station at a time, applying side BCs. Elliptic: every point depends on every other point, so all points are solved simultaneously, with BCs on the entire closed boundary.

52. α = k/ρc_v compares a fluid element's ability to conduct heat with its ability to store it (its heat capacity). Large α means faster spreading of temperature changes.

53. Parabolic: from the time derivatives of velocity and internal energy, as in the heat equation. Elliptic: from the viscous terms, which feed information upstream.

E · Worked solutions

54. $A = 1 - M^2,\ B = 0,\ C = 1$, so $D = -4(1 - M^2)$. For $M = 0.5$: $D = -3 < 0$ → **elliptic**. For $M = 2$: $D = 12 > 0$ → **hyperbolic**. Slopes: $\dfrac{dy}{dx} = \dfrac{\pm\sqrt{D}}{2A} = \dfrac{\pm\sqrt{12}}{2(-3)} = \mp\dfrac{1}{\sqrt{3}}$, i.e. $\pm\dfrac{1}{\sqrt{M^2 - 1}}$. Angle $= \tan^{-1}(1/\sqrt{3}) = 30^\circ = \sin^{-1}(1/M)$, the **Mach angle** μ.

55. $A = y,\ B = 0,\ C = 1$, so $D = -4y$. For $y > 0$: **elliptic**. For $y < 0$: **hyperbolic**. On $y = 0$: parabolic (the "sonic line"). The elliptic region ($y > 0$) plays the role of the subsonic zone, so this is a model of mixed (transonic) flow.

56. $\alpha = \dfrac{0.026}{1.2 \times 718} = \dfrac{0.026}{861.6} = 3.02 \times 10^{-5}\ \mathrm{m^2/s}$. Then $\dfrac{L^2}{\alpha} = \dfrac{(0.01)^2}{3.02 \times 10^{-5}} = \dfrac{10^{-4}}{3.02 \times 10^{-5}} \approx 3.3\ \mathrm{s}$. (Heat-transfer texts often use c_p in place of c_v. Follow Anderson's definition in this course.)

57. See Problem 3.2: $A = \alpha,\ B = C = 0$, so $D = 0$ → parabolic, with characteristics $t =$ const.

58. See Problem 3.5: the characteristic is $dx/dt = c$ (real) → hyperbolic. Compatibility gives $du = 0$, so u is constant along $x - ct =$ const, and $u = F(x - ct)$.

59. First: $D = 36 - 4(1)(9) = 0$ → **parabolic**. Second: $D = 1 - 4(3)(1) = -11 < 0$ → **elliptic**.

60. Start from $\rho\, De/Dt = \rho\dot{q} + \nabla\cdot(k\nabla T) - p\nabla\cdot\mathbf{V} + \Phi$. Assume (i) $\mathbf{V} = 0$ (stationary fluid), so $De/Dt = \partial e/\partial t$ and $p\nabla\cdot\mathbf{V} = \Phi = 0$, giving Eq. 3.25. (ii) $\dot{q} = 0$, and (iii) $e = c_v T$, giving Eq. 3.26. (iv) Constant k and ρc_v, giving Eq. 3.27, $T_t = \alpha\nabla^2 T$. (v) 1-D in x, giving Eq. 3.28, $T_t = \alpha T_{xx}$.

10 · One-Page Revision Sheet

CLASSIFY

$A\phi_{xx} + B\phi_{xy} + C\phi_{yy}$: $D = B^2 - 4AC$

> 0 hyperbolic · = 0 parabolic · < 0 elliptic

Characteristics: $\frac{dy}{dx} = \frac{B \pm \sqrt{D}}{2A}$

System: $|A| = 0$ gives the characteristics; $N = 0$ gives the compatibility equation.

PARABOLIC

1 characteristic (vertical line through P). Influences everything downstream between the boundaries. **March.**

2-D: IC on ac, BC on ab, cd. 3-D: IC on abcd (yz plane), BC on abgh, cdef, ahed, bgfc.

Flows: steady BL · PNS · unsteady conduction.

BOUNDARY LAYER

Prandtl 1904, Heidelberg. Thin layer, large Re = $\rho_\infty V_\infty L/\mu_\infty$. March in s.

IC: ab, ef from a self-similar wedge (2-D) or cone (axisymmetric) solution.

Wall ad, eh: no-slip. Edge bc, fg: inviscid surface solution.

PNS

Drop streamwise viscous derivatives, assume steady → parabolic.

+ simpler, + downstream marching. – no streamwise separation (upstream influence lost). Ideal for a viscous shock layer (Fig. 3.11).

HEAT CONDUCTION

3.25: $\rho e_t = \rho\dot{q} + \nabla\cdot(k\nabla T)$ $(\mathbf{V} = 0)$

3.26: $\dot{q} = 0,\ e = c_v T$

3.27: $T_t = \alpha\nabla^2 T,\ \alpha = k/\rho c_v$

3.28: $T_t = \alpha T_{xx}$; final steady profile is linear.

ELLIPTIC

Imaginary characteristics; no MOC. Closed domain; influence everywhere. **Simultaneous solution**: a "jury problem".

BC on the whole boundary: Dirichlet (values), Neumann (derivatives), mixed.

Flows: steady subsonic inviscid (M < 1) · incompressible inviscid ("queen", a → ∞, M = 0).

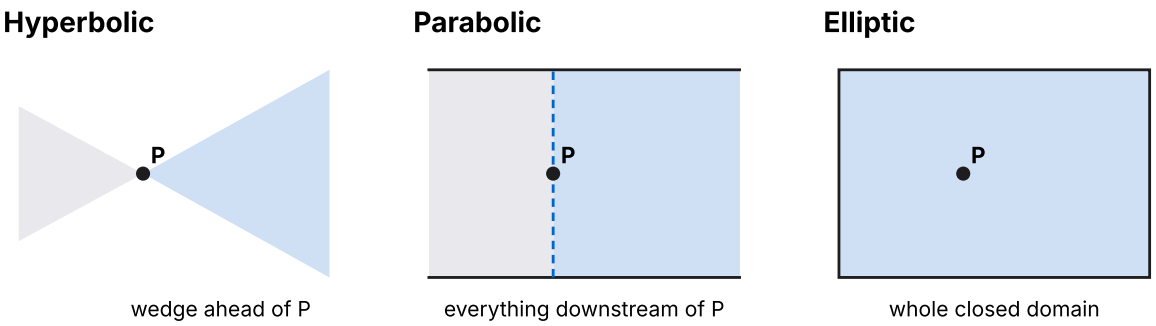
BLUNT BODY

Steady flow is mixed (elliptic nose / hyperbolic downstream), so it was ill-posed. Unsteady Euler is hyperbolic in t for any M → time-march from arbitrary IC → steady state. Moretti & Abbett 1966.

NS · WELL-POSED

Unsteady NS is mixed: parabolic (time terms) + elliptic (viscous terms). Time-marching is still well-posed.

Well-posed = exists + unique + continuous dependence on IC/BC.



wedge ahead of P | everything downstream of P | whole closed domain